AE3610 Experiments in Fluid and Solid Mechanics

COLUMN BUCKLING MEASUREMENTS

Objective
This primary object of this experiment is to explore the performance of columns and beam-columns. The experiment includes two tests: a long column and a short column. Each will be tested to explore the change in critical buckling strength created by changes in geometry and loading conditions. In addition, students will gain familiarity with buckling analysis techniques such as Southwell plots and the Euler hyperbola.

Background
Slender structures subject to compressive loads often exhibit elastic instabilities. One type of elastic stability is known as buckling. Aerospace structures are often slender and are subject to compressive loads, therefore, understanding structural instability in general and buckling in particular is required for aerospace structural design.

Structural instability is a fundamentally nonlinear phenomena, therefore to predict buckling we must model the structural nonlinearity. The most common structural instabilities are due to geometric nonlinearity rather than nonlinear material behavior. Geometric nonlinearity accounts for the effect of deformation within an equilibrium analysis rather than assuming infinitesimal structural deflections.

Buckling of a Perfect Column (Euler Buckling)
Consider the buckling of a column loaded by opposing axial loads as shown in Figure 1. We can model this using an extension of Euler-Bernoulli beam theory. Using this theory, the transverse deformation, \(w(x)\), of a beam is governed by the equation (1)

\[
EI \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = q(x)
\]

where \(E\) and \(I\) are the elastic modulus and second moment of area, \(P\) is the axial compressive load and \(q(x)\) is the distributed load. At first glance, this equation may not appear to be nonlinear, however, the second term \(P \left( \frac{d^2 w}{dx^2} \right)\) is, in fact, a nonlinear term. If we consider
a beam that is only subject to a compressive load, with \( q(x) = 0 \), then the governing equation can be rewritten as

\[
\frac{d^4 w}{dx^4} + k^2 \frac{d^2 w}{dx^2} = 0
\]

where \( k^2 = P / EI \).

Note that the governing equation (2) is now an eigenproblem. Using simply supported boundary conditions for the ends of the beam, we find the eigenvalues of (2) are given by equation (3).

\[
k_n L = n\pi \quad \text{for } n=1,2,3,\ldots
\]

Rearranging this expression gives the following equation for the loads corresponding to these eigenvalues

\[
P_n = n^2 \frac{\pi^2 EI}{L^2}.
\]

We are most interested in the lowest load for which the beam becomes unstable, i.e., the critical load. This corresponds to the case with \( n = 1 \), i.e.,

\[
P_{cr} = \frac{\pi^2 EI}{L^2}.
\]

The critical load based on this theory is often referred to as the Euler buckling load. Using this load, we can compute the axial stress in the beam when it buckles

\[
\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{L^2 A}
\]

where \( A \) is the cross-sectional area of the beam. This critical stress is often written in terms of the radius of gyration \( (R_g) \) of the column about its weak axis, which is defined as \( R_g = \sqrt{I/A} \).

With this definition, the critical buckling load stress and strain from this theory are given by

\[
\sigma_{cr} = \left( \frac{\pi R_g}{L} \right)^2 E
\]

\[
\varepsilon_{cr} = \left( \frac{\pi R_g}{L} \right)^2.
\]

**Buckling of an Imperfect Column and the Southwell Plot**

The model of Euler buckling is flawed. In particular, the beam is not perfectly straight and the axial compressive load is not applied through the center of the beam. As a result, the beam
does not suddenly buckle at $P_{cr}$ but more gradually buckles in a direction determined by the imperfections in the beam. Such an imperfect beam is shown in Figure 2. Even though these imperfections can be quite small, they can have a large effect on the response of the structure. This is called imperfection sensitivity.

A more sophisticated analysis that takes into account an initial imperfection of the beam, where the imperfection causes an initial displacement with the form $a_i \sin(\pi x/L)$, results in the following expression for the transverse deflection of the beam at its mid-point ($x=L/2$) as a function of the Euler buckling load.

$$\delta = \frac{Pa_i}{P_{cr} - P}.$$  \hspace{1cm} (8)

This relationship forms the basis of the Southwell plot. Rearranging this relationship, one can create an expression for the deflection normalized by the load, i.e.,

$$\frac{\delta}{P} = \frac{\delta}{P_{cr}} + \frac{a_i}{P_{cr}}.$$  \hspace{1cm} (9)

A Southwell plot consists of a series of measurements plotted on a graph of $\delta/P$ versus $\delta$. The slope of a linear fit to the data then provides $P_{cr}$ and the $y$-intercept provides a measure of the initial displacement magnitude (see Figure 3a).

In this lab, however, we will only be able to measure the bending strains using the strain gages mounted to the specimen. Therefore, we will modify the classical Southwell plot to use our strain measurements as follows. First, we note that the bending moment at the mid-point of the beam is $M = \delta P$. With the expression for the bending strain, $\varepsilon_{\text{bend}} = -yM/EI$, the deflection of the beam at the mid-span is

$$\delta = \frac{2EI}{Ph} \varepsilon_{\text{bend}}.$$  \hspace{1cm} (10)

where we have used $y = -h/2$ as the bending moment. Substitution of this value for $\delta$ into (9) yields

$$\varepsilon_{\text{bend}} = \frac{\varepsilon_{\text{bend}}}{P} P_{cr} - a_i \frac{Ph}{2EI}.$$  \hspace{1cm} (11)

This version of the Southwell plot for strain (see Figure 3b) is the one you will use for data reduction in this lab.
Euler’s Hyperbola

In order to compare measurements to theory for different materials and column geometries, it is common to construct what is called Euler’s Hyperbola. This is a plot of the critical load stress $\sigma_{cr}$ versus the slenderness ratio ($s \equiv L/R_g$). In this way, both the column geometry (e.g., $L, I$) and material properties (i.e., $E$) are taken into account in a single curve. On such a curve the critical stress approaches infinity as the slenderness ratio gets small (i.e., for short columns). The practical upper limit for using such a graph is based on the yield stress of the material; when $\sigma_{cr} \geq \sigma_y$, the column will fail inelastically rather than by elastic buckling.

Measurements

All the test subjects are composed of 2024-T3 aluminum. Each column will only be tested in the elastic regime to ensure accurate results with the strain gages and so as not to permanently deform the columns. There are two test specimens: a long and a short column; both have rectangular cross-sections (see Table 1 for dimensions).

When pure axial loads are applied to the nominally perfect columns, we have no advance knowledge of any preferential buckling direction; thus it is difficult to correctly place any lateral displacement gage. As a result, the columns are fitted with two strain gages at mid-span mounted on the top and bottom surfaces of the beam. Assuming that the strain varies linearly through the beam, the axial strain measured at the top ($e_a$) and bottom ($e_b$) surfaces of the beam would be

$$e_a = e_{\text{axial}} + e_{\text{bend}}$$
$$e_b = e_{\text{axial}} - e_{\text{bend}}$$

where $e_{\text{axial}}$ is the strain from the axial deformation and $e_{\text{bend}}$ is the strain due to bending. To find these two strains, we could combine the measured strains to produce

$$e_{\text{axial}} = \frac{1}{2} (e_a + e_b)$$
$$e_{\text{bend}} = \frac{1}{2} (e_a - e_b).$$

Due to the fact that the direction of buckling is unknown, the strain reading may either be positive or negative.

Note that the bending strain is also given by
\[ \varepsilon_{\text{bend}} = -\frac{h}{2} \kappa = -\frac{hM}{2EI}. \] (14)

**Preliminary**

*The following items must be turned in at the start of your lab session.*

1. A choice of the loads to be used for each test, i.e., for Steps 3, 4, 6 and 7 in the Procedure.

**Procedure**

1. Locate the test fixture to be used to connect the beams to the Instron Load Frame. The fixture consists of two small round vee-grooved inserts that will be located in the small holes in the two aluminum plates attached to the lower crosshead and upper load cell extension in the Instron Load Frame.

**Long Column Tests**

2. Mount the long column between the vee-groove adapters.

3. Load the column under manual load control from 100lb to 800lb. Use five different load settings. For each setting record load and bending strain readings.

4. Again using manual load control, record the load and bending strains as you unload the beam. For the unloading run, use five loadings; repeat one of the intermediate loadings from Step 3, but choose different loadings for the other four points.

**Short Column Tests**

5. Remove the long column and mount the short column between the vee-groove adapters.

6. Load the column under manual load control from 500lb to 1100lb. Use five different load settings. For each setting record load and bending strain readings.

7. Using manual load control, record the load and bending strains as you unload the beam; again, repeat one of the intermediate loadings from Step 6, but choose different loadings for the other four points.

**Data to be Taken**

1. Axial loads and bending strains for loading and unloading runs for long column.

2. Axial loads and bending strains for loading and unloading runs for short column.
Data Reduction

1. Using a Southwell plot for each of the columns, determine the column’s buckling load and buckling (axial) stress.

2. Calculate the theoretical buckling load and stress for each column based on beam theory.

Results Needed for Report

1. Table of measured strains for each of the applied (measured) axial loads for long column.
2. Table of measured strains for each of the applied (measured) axial loads for short column.
3. Individual Southwell plots for each of the columns.
4. Table of experimentally determined and predicted buckling loads for each column.
5. Euler’s hyperbola plot containing both the experimentally determined and predicted data.

Table 1. Dimension of aluminum columns.

<table>
<thead>
<tr>
<th></th>
<th>Long Column</th>
<th>Short Column</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (in.)</td>
<td>29</td>
<td>24</td>
</tr>
<tr>
<td>Width (in.)</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>Thickness (in.)</td>
<td>0.50</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Figure 1. Euler-Bernoulli beam subject to a compressive load, $P$. 
Figure 2. Imperfect Euler beam with initial curvature.

Figure 3. Southwell plots for determining the critical load: (a) Southwell plot in original form, (b) Southwell plot for strain. The figures are not drawn to scale, and the slopes on the two graphs are the reciprocal of one another.