Experimental Errors and Uncertainty: An Introduction

Prepared for students in AE 2610

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adapted from material by J. Craig

Outline

• Errors and types of error
• Statistic/probability: confidence levels
• Uncertainty analysis

Experimental Error

• Error: all measurements have some uncertainty
  error = \varepsilon = u_{\text{meas}} - u_{\text{exact}}

• Objectives
  1. Minimize error so that
     \[-\Delta \leq \varepsilon \leq +\Delta\] within some uncertainty (statistical confidence)
     or \[u_{\text{meas}} - \Delta \leq u_{\text{exact}} \leq u_{\text{meas}} + \Delta\]
  2. Estimate error (uncertainty) to
determine reliability, meaningfulness of data
Accuracy and Precision

- **Accuracy**: also called *systematic or bias error*
  - denotes something repeatably “wrong” with the measurement or experiment
- **Precision**: also called *random error or noise*
  - denotes errors that change randomly each time you try to repeat experiment

![Errors and types of error](AE2610)

Some Systematic Measurement Errors

- $u = u(x)$

  - **Nonzero offset - Background**
  - **Backlash & Hysteresis**
  - **Drift** (e.g., offset changing in systematic way with time)

- **Systematic errors can be eliminated/removed if they are known**

![Errors and types of error](AE2610)
Some Random Measurement Errors

- After data acquired, nearly impossible to separate random error (noise) sources
- Examine random error with statistical methods

Uncertainty – Probability and Statistics

- We do not know the exact error
  - if we did, we would correct it and be error free
- We must estimate the error or uncertainty in our measurement
  - this requires application of some basic probability and statistics
Probability Distributions

- When we make measurements (i.e., take samples) of a system a number of times, we will get a distribution of results.

- We might even make just one measurement (sample) of a system that has a distribution of possible states.

Statistics and probability

- Since we can not make an infinite number of measurements to determine the true probability distribution:
  - we use statistics to make estimates based on assumed distribution function.

- Some useful analytic distribution functions:
  - normal (Gaussian)
    - student’s t
    - log normal
    - exponential

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]
Normal Distributions – Probability Range

- What fraction of values (combined probability) lie within a given range from mean for a normal distribution?

\[
\begin{align*}
\text{One Sigma:} & \quad \text{Prob}(\mu - \sigma \leq x \leq \mu + \sigma) = \int_{\mu - \sigma}^{\mu + \sigma} f(x) \, dx = 0.683 \\
\text{Two Sigma:} & \quad \text{Prob}(\mu - 2\sigma \leq x \leq \mu + 2\sigma) = \int_{\mu - 2\sigma}^{\mu + 2\sigma} f(x) \, dx = 0.954 \\
\text{Three Sigma:} & \quad \text{Prob}(\mu - 3\sigma \leq x \leq \mu + 3\sigma) = 0.997
\end{align*}
\]

Sample Statistics

- What if \( \mu \) and \( \sigma \) are unknown (as is often the case)?
  - use estimates from measurements, \( \bar{x} \) and \( s_x \)

\[
\begin{align*}
\mu & \approx \bar{x}; \\
\text{Sample mean} = \bar{x} & = \frac{1}{N} \sum_{i=1}^{N} x_i \\
\text{Sample variance} = s_x^2 & = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2
\end{align*}
\]
Uncertainty Estimates

• Question: If one takes $N$ (large) readings and computes $\bar{x}$, how confident can you be that the average is really close to the true mean ($\mu$)?

• Confidence intervals are way to describe this

$$\text{Prob} = c\% \text{ that } \mu \text{ lies in shaded area defined by } \bar{x} = \mu \pm a_x \sigma_{\bar{x}}$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} \approx \frac{s}{\sqrt{N}}$$

Confidence Levels

• For normal distribution

<table>
<thead>
<tr>
<th>Error Level Name</th>
<th>Error Level</th>
<th>Prob. that Error is Smaller</th>
<th>Prob. that Error is Larger</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probable Error</td>
<td>$\pm 0.67 \sigma$</td>
<td>50%</td>
<td>1:2</td>
</tr>
<tr>
<td>One Sigma</td>
<td>$\pm \sigma$</td>
<td>68%</td>
<td>~1:3</td>
</tr>
<tr>
<td>90% error</td>
<td>$\pm 1.64 \sigma$</td>
<td>90%</td>
<td>1:10</td>
</tr>
<tr>
<td>&quot;Two&quot; Sigma</td>
<td>$\pm 1.96 \sigma$</td>
<td>95%</td>
<td>1:20</td>
</tr>
<tr>
<td>Three Sigma</td>
<td>$\pm 3 \sigma$</td>
<td>99.7%</td>
<td>1:370</td>
</tr>
<tr>
<td>Maximum Error</td>
<td>$\pm 3.29 \sigma$</td>
<td>99.9%</td>
<td>1:1000</td>
</tr>
<tr>
<td>Four Sigma</td>
<td>$\pm 4 \sigma$</td>
<td>99.994%</td>
<td>1:10000</td>
</tr>
<tr>
<td>Six Sigma</td>
<td>$\pm 6 \sigma$</td>
<td>99.99999999%</td>
<td>1:1.01e9</td>
</tr>
</tbody>
</table>

Example:

$$c = 95\% \Rightarrow a_x = 1.96, \text{ so } \bar{x} = \mu \pm 1.96 \sigma_{\bar{x}}$$

With 95% confidence $\mu$ will fall within $\bar{x} \pm 1.96 \sigma_{\bar{x}}$
Example 1

• You have a differential pressure transducer you are using to measure the q of a wind tunnel. The manufacturer says that the transducer is linear to within 0.05% of full-scale (10 Torr). You measure the q 2500 times and find the average is 1.5 Torr and the rms is 0.05 Torr.
• What are the systematic and random uncertainties of your measurement?
• Systematic
  – \( u_{\text{syst}} = 0.05\% \times 10 \text{ Torr} = 0.005 \text{ Torr} \)
• Random
  – \( s_x/N^{0.5} = 0.05 \text{ Torr} / 2500^{0.5} = 0.001 \text{ Torr} \)
  – using a 95% confidence level,
    \( u_{\text{rand}} = 1.96 \times 0.001 \text{ Torr} = 0.002 \text{ Torr} \)

Combining Bias and Precision Uncertainties

• We noted earlier that errors in each measured variable (\(y_i\)) will include both bias (systematic) and precision (random) components
• These can usually be treated as independent and therefore the uncertainties for each can be combined into a total:
  \[ u_{y_i-\text{total}} = \left[ u_{y_i-\text{precision}}^2 + u_{y_i-\text{bias}}^2 \right]^{1/2} \]
  – note: if they are not independent, combining in other ways may be necessary
• In terms of confidence intervals
  \[ u_{y_i-\text{total}} = \left[ (a_i \sigma_{y_i})^2 + u_{y_i-\text{bias}}^2 \right]^{1/2} \]
Example 1 - Continued

• So what is the combined (systematic and random) uncertainty of our measurement?
  
  \[ u_{\text{systematic}} = 0.005 \text{ Torr} \]
  
  \[ u_{\text{random}} = 0.002 \text{ Torr} \text{ (95\% confidence)} \]
  
  \[ u_{\text{total}} = (0.005^2 + 0.002^2)^{1/2} \text{ Torr} = 0.0054 \text{ Torr} \]

• So we would report a mean pressure of \( 1.5 \pm 0.0054 \text{ Torr} \)

Analysis of Combined Uncertainties

• Many times experimental results are the result of several independent measurements combined using a theoretical formula (e.g., \( F = m \frac{v_2 - v_1}{\Delta t} \) for the force of acceleration on a body)

• What is the uncertainty of the mean of a value if we compute that value from the means of various measurements?

  \[ u_y = ? \text{ if } y = y(x_1, x_2, \ldots, x_N) \]

For uncertainties, \( u_i \), that are small compared to \( x_i \), we can use:

\[ u_y = \left[ \left( \frac{\partial y}{\partial x_1} u_1 \right)^2 + \left( \frac{\partial y}{\partial x_2} u_2 \right)^2 + \cdots + \left( \frac{\partial y}{\partial x_N} u_N \right)^2 \right]^{1/2} \]
Uncertainty Example

- We are measuring force indirectly
  - measured mass of object, velocity of object at time 1, velocity of object at time 2

<table>
<thead>
<tr>
<th></th>
<th>Mass (kg)</th>
<th>Velocity 1 (m/s)</th>
<th>Velocity 2 (m/s)</th>
<th>( \Delta t ) (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>10.1</td>
<td>31.5</td>
<td>34.2</td>
<td>55.1</td>
</tr>
<tr>
<td>RMS</td>
<td>0.1</td>
<td>0.32</td>
<td>0.33</td>
<td>0.1</td>
</tr>
<tr>
<td>Number of samples</td>
<td>10</td>
<td>900</td>
<td>900</td>
<td>900</td>
</tr>
</tbody>
</table>

- Find: Accelerating force based on the means of our measurements, including the uncertainty in our “measured” force

\[
F = m \frac{v_2 - v_1}{\Delta t}
\]

\[
u_T = \left( \left( \frac{\partial F}{\partial m} \right) u_m \right)^2 + \left( \left( \frac{\partial F}{\partial v_2} \right) u_{v_2} \right)^2 + \left( \left( \frac{\partial F}{\partial v_1} \right) u_{v_1} \right)^2 + \left( \left( \frac{\partial F}{\partial \Delta t} \right) u_{\Delta t} \right)^2 \]

\[
= \left( \frac{v_1}{\Delta t} \right)^2 + \left( \frac{v_2}{\Delta t} \right)^2 + \left( \frac{m}{\Delta t} \right)^2 + \left( \frac{m}{\Delta t} \right)^2 + \left( \frac{m(v_2 - v_1)}{(\Delta t)^2} \right)^2
\]

\[
\Delta F = 6.3
\]

- so the accelerating force we measure is 
  \(495 \pm 6.3\) N with 95% confidence

- the largest sources of uncertainty here are the velocity measurements (due to the subtraction in the expression)
Can have error bars in vertical and/or horizontal coordinates.