Experimental Errors and Uncertainty: An Introduction

Prepared for students in AE 3051

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adapted from material by J. Craig

Outline

• Errors and types of error
• Statistic/probability: confidence levels
• Uncertainty analysis
Experimental Error

• **Error**: all measurements have some uncertainty
  \[
  \text{error} = \varepsilon = u_{\text{meas}} - u_{\text{exact}}
  \]

• **Objectives**
  1. **Minimize** error so that
     \[-\Delta \leq \varepsilon \leq +\Delta \text{ within some uncertainty (statistical confidence)}
     \]
     or \[u_{\text{meas}} - \Delta \leq u_{\text{exact}} \leq u_{\text{meas}} + \Delta\]
  2. **Estimate** error (uncertainty) to
deretermine reliability, meaningfulness of data

\[\text{Errors and types of error} \]
Accuracy and Precision

• **Accuracy**: also called *systematic* or *bias error*
  – denotes something repeatably “wrong” with the measurement or experiment

• ** Precision**: also called *random error* or *noise*
  – denotes errors that change randomly each time you try to repeat experiment
Accuracy/Systematic Errors

• Sources
  – Measuring system errors
    • difference between model of measuring system and reality
    • could be corrected, e.g., with better model of measurement
  – Measured system “errors”
    • influence of uncontrolled or unaccounted for variables in the experiment
    • the measured data may be “correct”, but may lead to an incorrect model of the object/process being studied
  – Blunders
    • human errors - misunderstandings
Some Systematic Measurement Errors

- $u = u(x)$

- **Nonzero offset - Background**

- **Backlash & Hysteresis**

- **Drift** (e.g., offset changing in systematic way with time)

- **Model**

- **Actual**

**Nonlinearity**

**Quantization Error** (digitized data – impacts resolution)

- Systematic errors can be eliminated/removed if they are known

Errors and types of error

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Some Random Measurement Errors

- Background “Noise” (offset changing randomly with time)
- Detector “Noise” (random change in sensitivity of device)

After data acquired, nearly impossible to separate random error (noise) sources

Examine random error with statistical methods
Other Related Terms

• Sensitivity
  – Change in a measurement device’s output for a unit change in the measured (input) quantity, e.g., volts/Torr for the Baratron

• Resolution
  – Smallest increment of change in a system or property that a measurement device can reliably capture

• Dynamic Range
  – Maximum output of a measurement device divided by its resolution (or minimum measurable signal)

Sensitivity
\[
\text{Sensitivity} = \frac{5000 \text{ psi}}{270^\circ} = 18.5 \text{ psi/degree}
\]

Resolution
\[
\text{Resolution} = 50 \text{ psi}
\]

Dynamic Range
\[
\text{Dyn. Range} = \frac{5000}{50} = 100
\]
Uncertainty – Probability and Statistics

• We do not know the exact error
  – if we did, we would correct it and be error free

• We must estimate the error or uncertainty in our measurement
  – this requires application of some basic probability and statistics
Probability Distributions

- When we make measurements (i.e., take samples) of a system a number of times, we will get a distribution of results.

- We might even make just one measurement (sample) of a system that has a distribution of possible states.

\[ \int_{-\infty}^{\infty} f(y) dy = 1 \]
Statistics and Probability

• Since we can not make an infinite number of measurements to determine the true probability distribution
  – we use statistics to make estimates based on assumed distribution function

• Some useful analytic distribution functions
  – normal (Gaussian)
  – student’s t
  – log normal
  – exponential
Commonly used when measurements/measurement system:

- made up from many independent systems, each with any kind of distribution
- # samples taken is very large (e.g., sample means)
- more…

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[ -\frac{(x - \mu)^2}{2\sigma^2} \right] \]

\( \mu = \text{mean} \)
\( \sigma^2 = \text{variance} \)
\( \sigma = \text{standard deviation} \)

Normal/Gaussian Probability Distribution
Normal Distributions – Probability Range

- What fraction of values (combined probability) lie within a given range from mean for a normal distribution?

One Sigma: \( \text{Prob}(\mu - \sigma \leq \mu \leq \mu + \sigma) = \int_{\mu - \sigma}^{\mu + \sigma} f(x) \, dx = 0.683 \)

Two Sigma: \( \text{Prob}(\mu - 2\sigma \leq \mu \leq \mu + 2\sigma) = \int_{\mu - 2\sigma}^{\mu + 2\sigma} f(x) \, dx = 0.954 \)

Three Sigma: \( \text{Prob}(\mu - 3\sigma \leq \mu \leq \mu + 3\sigma) = 0.997 \)
Sample Statistics

• What if \( \mu \) and \( \sigma \) are unknown (as is often the case)?
  – use estimates from measurements, \( \bar{x} \) and \( s_x \)

\[
\mu \cong \bar{x}; \; \sigma \cong s_x
\]

\[
\text{Sample mean} = \bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}
\]

\[
\text{Sample variance} = s_x^2 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N - 1} = \left( \frac{\sum_{i=1}^{N} x_i^2}{N} - \bar{x}^2 \right) \frac{N}{N - 1}
\]

  – use \((N-1)\) for \(s_x^2\) because we have \(N\) independent \(x_i\) but if
also know \(x_{\text{mean}}\) then only need to know \((N-1)\) \(x_i\) to
compute last remaining \(x_i\)

\[\Rightarrow \text{only } (N-1) \text{ “degrees of freedom” for this calculation}\]
Uncertainty Estimates

• Question: If one takes $N$ (large) readings and computes $\bar{x}$, how confident can you be that the average is really close to the true mean ($\mu$)?

• Confidence intervals are way to describe this

$$\text{Prob} = c\% \text{ that } \mu \text{ lies in shaded area defined by } \bar{x} = \mu \pm a_c \sigma_{\bar{x}}$$

where $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} \approx \frac{S_x}{\sqrt{N}}$

$a_c =$?
Confidence Levels

- For normal distribution

<table>
<thead>
<tr>
<th>Error Level Name</th>
<th>Error Level</th>
<th>Prob. that Error is Smaller</th>
<th>Prob. that Error is Larger</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probable Error</td>
<td>± 0.67 σ</td>
<td>50%</td>
<td>1:2</td>
</tr>
<tr>
<td>One Sigma</td>
<td>± σ</td>
<td>68%</td>
<td>~ 1:3</td>
</tr>
<tr>
<td>90% error</td>
<td>± 1.65 σ</td>
<td>90%</td>
<td>1:10</td>
</tr>
<tr>
<td>“Two” Sigma</td>
<td>± 1.96 σ</td>
<td>95%</td>
<td>1:20</td>
</tr>
<tr>
<td>Three Sigma</td>
<td>± 3 σ</td>
<td>99.7%</td>
<td>1:370</td>
</tr>
<tr>
<td>Maximum Error</td>
<td>± 3.29 σ</td>
<td>99.9%</td>
<td>1:1000</td>
</tr>
<tr>
<td>Four Sigma</td>
<td>± 4 σ</td>
<td>99.994%</td>
<td>1:16000</td>
</tr>
<tr>
<td>Six Sigma</td>
<td>± 6 σ</td>
<td>99.99999999%</td>
<td>1:1.01e9</td>
</tr>
</tbody>
</table>

Example:

\[ c = 95\% \Rightarrow a_c = 1.96, \text{ so } \bar{x} = \mu \pm 1.96 \sigma_{\bar{x}} \]

With 95% confidence, \( \mu \) will fall within \( \bar{x} \pm 1.96 \sigma_{\bar{x}} \)
Example 1

You have a differential pressure transducer you are using to measure the q of a wind tunnel. The manufacturer says that the transducer is linear to within 0.05% of full-scale (10 Torr). You measure the q 2500 times and find the average is 1.5 Torr and the rms is 0.05 Torr.

What are the systematic and random uncertainties of your measurement?

- **Systematic**
  - \( u_{\text{syst}} = 0.05\% \text{ of } 10 \text{ Torr} = 0.005 \text{ Torr} \)

- **Random**
  - \( s_x/N^{0.5} = 0.05 \text{ Torr}/2500^{0.5} = 0.001 \text{ Torr} \)
  - using a 95% confidence level,
    \( u_{\text{rand}} = 1.96 \times 0.001 \text{ Torr} = 0.002 \text{ Torr} \)
Combining Bias and Precision Uncertainties

• We noted earlier that errors in each measured variable \( (y_i) \) will include both bias (systematic) and precision (random) components.

• These can usually be treated as independent and therefore the uncertainties for each can be combined into a total:

\[
\sigma_{y_i-\text{total}} = \left( \sigma_{y_i-\text{precision}}^2 + \sigma_{y_i-\text{bias}}^2 \right)^{1/2}
\]

- note: if they are not independent, combining in other ways may be necessary

• In terms of confidence intervals

\[
\sigma_{y_i-\text{total}} = \left[ (a_c \sigma_{\bar{y}_i})^2 + \sigma_{y_i-\text{bias}}^2 \right]^{1/2}
\]
Analysis of Combined Uncertainties

- Many times experimental results are the result of several independent measurements combined using a theoretical formula (e.g., \( \dot{m} = \rho u A = \frac{P}{RT} \pi D^2 / 4 \) for mass flowrate through a pipe).

- How do uncertainties in each variable contribute to whole?

If \( y = y(x_1, x_2, \ldots, x_N) \) is a linear function, a statistical theorem states that:

\[
\sigma_y = \left[ \left( \frac{\partial y}{\partial x_1} \sigma_{x_1} \right)^2 + \left( \frac{\partial y}{\partial x_2} \sigma_{x_2} \right)^2 + \cdots + \left( \frac{\partial y}{\partial x_N} \sigma_{x_N} \right)^2 \right]^{1/2}
\]

For uncertainties, \( u_i \), that are small compared to \( x_i \) we can use a Taylor Series expansion in \( u_i \):

\[
y(x_1 + u_1, x_2 + u_2, \ldots, x_N + u_N) = y(x_1, x_2, \ldots, x_N) + \frac{\partial y}{\partial x_1} u_1 + \frac{\partial y}{\partial x_2} u_2 + \cdots + \frac{\partial y}{\partial x_N} u_N
\]

so that \( y \) is now a linear function of the uncertainties. Applying this in the first equation yields:

\[
u_y = \left[ \left( \frac{\partial y}{\partial x_1} u_1 \right)^2 + \left( \frac{\partial y}{\partial x_2} u_2 \right)^2 + \cdots + \left( \frac{\partial y}{\partial x_N} u_N \right)^2 \right]^{1/2}
\]
Example: Uncertainty Calculation

- Consider measurement of mass flowrate through a round pipe

\[
m = \rho v A = \frac{p}{RT} \sqrt[4]{\frac{\pi}{4}} D^2 \quad \Rightarrow \quad \bar{m} = \frac{\bar{p}}{RT} \frac{\bar{L}}{\Delta t} \frac{\pi}{4} D^2
\]

- The fractional uncertainty in \( \bar{m} \) can then be shown to be:

\[
\frac{u_{\bar{m}}}{\bar{m}} = \left[ \left( \frac{u_p}{p} \right)^2 + \left( \frac{u_T}{T} \right)^2 + \left( \frac{u_L}{L} \right)^2 + \left( \frac{u_{\Delta t}}{\Delta t} \right)^2 + \left( \frac{2 u_D}{D} \right)^2 \right]^{1/2}
\]

* e.g.,

\[
y = f \left( \frac{\partial y}{\partial x_1} u_1, \ldots \right) \quad \left( \frac{\partial \bar{m}}{\partial p} \right) u_p = \left( \frac{1}{RT} \frac{L}{\Delta t} \frac{\pi}{4} D^2 \right) u_p = \left( \frac{\bar{m}}{p} \right) u_p = \frac{u_p}{p} \bar{m}
\]

Uncertainty analysis
## Example: Uncertainty Calculation (cont’d)

Assume the following uncertainties \( (u_x/\text{precision} = s_x/N^{1/2}) \)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Accuracy ((u_x/x))</th>
<th>Precision ((u_x/x))</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>0.4%</td>
<td>0.1%</td>
<td>Pressure transducer with 8-bit digitizer</td>
</tr>
<tr>
<td>(T)</td>
<td>2%</td>
<td>0.4%</td>
<td>Temp. transducer with 1% full-scale linearity error used at half-scale</td>
</tr>
<tr>
<td>(\Delta t)</td>
<td>0.01%</td>
<td>2%</td>
<td>Accurate clock, but starting/stopping uncertainty of 0.01 sec for 0.5 sec measurement</td>
</tr>
<tr>
<td>(L)</td>
<td>0.1%</td>
<td>—</td>
<td>Only measured once with ruler having maximum 0.5 mm reading error over 0.5 m pipe length</td>
</tr>
<tr>
<td>(D)</td>
<td>1%</td>
<td>—</td>
<td>Only measured once with with ruler having maximum 0.5 mm reading error over 0.05 m diameter</td>
</tr>
<tr>
<td>Summed ((\Sigma u^2)^{1/2})</td>
<td>2.5%</td>
<td>2.0%</td>
<td>(\Delta t) meas. dominates precision error (D) meas. dominates accuracy error</td>
</tr>
</tbody>
</table>

95% Confidence

| 2.5% | 4.0% |

With 95% confidence level precision error dominant

Total uncertainty in a overall measurement of \(\dot{m}\) is then:

\[
\frac{u_{\dot{m}}}{\dot{m}} = \left[ u_{\text{accuracy}}^2 + u_{\text{precision}}^2 \right]^{1/2} = 4.7\%
\]

**Uncertainty analysis**
Plotting Uncertainty - Error Bars

• Can have error bars in vertical and/or horizontal coordinates

Uncertainty analysis